Finite Math - Fall 2018
Lecture Notes - 11/29/2018

Homework

- Section 5.3 - 1, 2, 5, 6, 9, 17, 19, 21, 23, 25, 27, 35

Section 5.3 - Linear Programming in Two Dimensions: A Geometric Approach

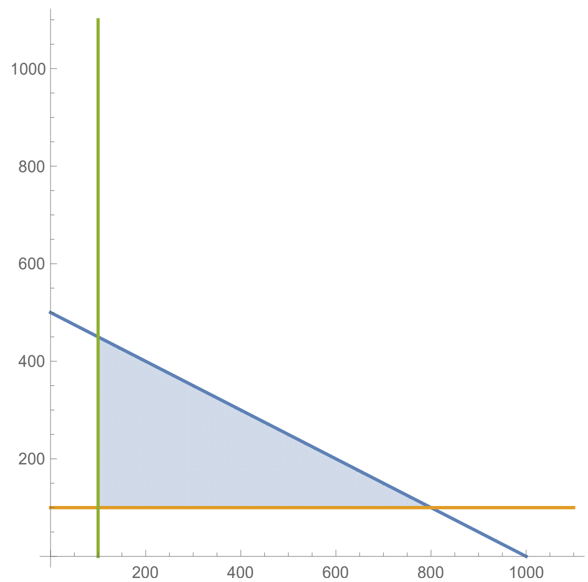
A Simple Linear Programming Problem.

Example 1. *A food vendor at a rock concert sells hot dogs for* \$4 *each and hamburgers for* \$5 *each. She purchases hot dogs for* 50¢ *each and hamburgers for* \$1 *each. If she has* \$500 *to spend on supplies, and wants to bring at least* 100 *each of hot dogs and hamburgers, how many hot dogs and hamburgers should she buy to make the most money at the concert? (Assume she sells her entire inventory.) What is her maximum revenue?*

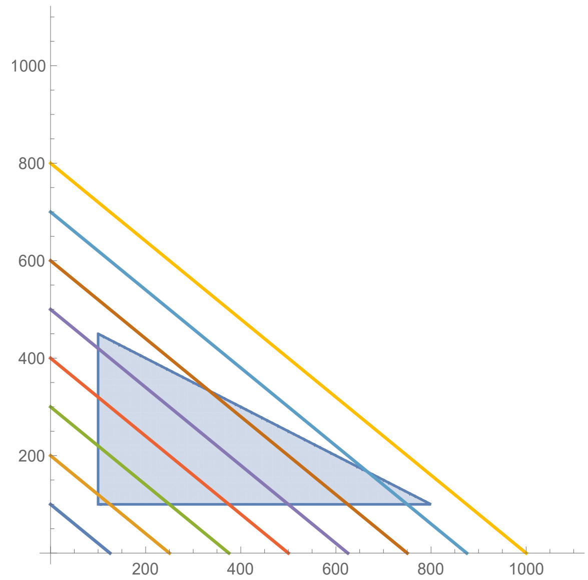
Solution. *Let x be the number of hot dogs and y be the number of hamburgers. Her revenue function is*

$$R = 4x + 5y$$

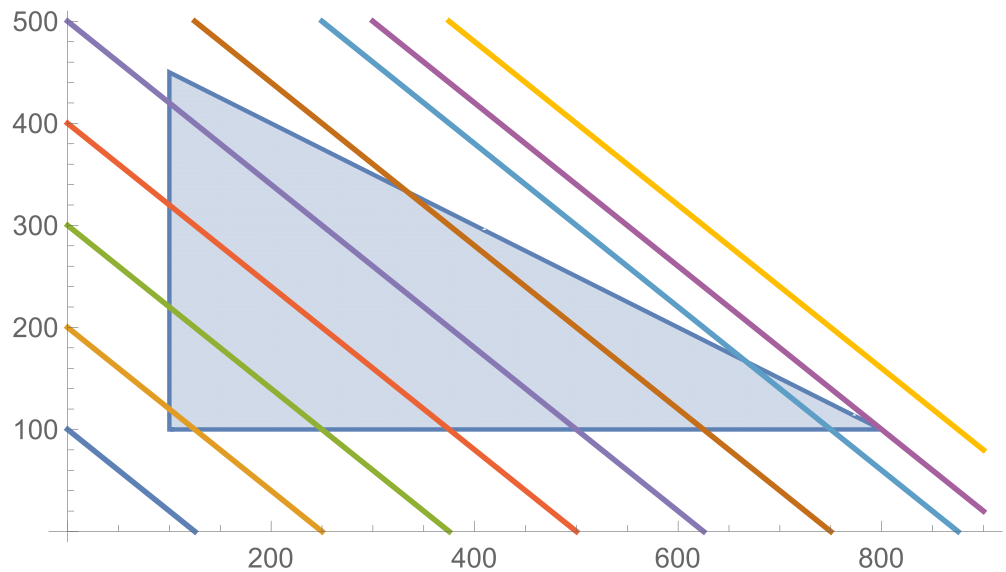
and since she is going to spend at most \$500 *on supplies, we get $0.5x + y \leq 500$. Also, since she will buy at least* 100 *each of hot dogs and hamburgers, we also have $x \geq 100$ and $y \geq 100$. Now, we graph the feasible region.*



To figure out if she can make R_0 dollars in sales, we graph the line $4x + 5y = R_0$ and see if it hits the feasible region. If it does, it is possible to make that much in sales. In the next figure, we graph the revenue function for the values $500, 1000, 1500, 2000, 2500, 3000, 3500, 4000$.



We know now that the maximum revenue is somewhere between \$3500 *and* \$4000. *If we observe how the intersection of the revenue line and the feasible region is changing, we can see that the last time the revenue line will hit the feasible region is at the corner point at the bottom right of the feasible region. Thus, the maximum revenue will happen if she sells* 800 *hot dogs and* 100 *hamburgers. Her revenue will be* $4(800) + 5(100) = 3700$ *dollars in this case. Observe if the line* $4x + 5y = 3700$ *is graphed in the above plot*



we can see that bumping that line slightly up/right will move it off of the feasible region, so the point $(800, 100)$ *must be the maximum, and the maximum revenue must be* \$3700.

General Description of Linear Programming. In a *linear programming problem*, we are concerned with *optimizing* (finding the maximum and minimum values, called the *optimal values*) of a linear *objective function* z of the form

$$z = ax + by$$

where a and b are not both zero and the *decision variables* x and y are subject to *constraints* given by linear inequalities. Additionally, x and y must be nonnegative, i.e., $x \geq 0$ and $y \geq 0$.

The following theorems give us information about the solvability and solution of a linear programming problem:

Theorem 1 (Fundamental Theorem of Linear Programming). *If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one or more of the corner points of the feasible region.*

Theorem 2 (Existence of Optimal Solutions).

(A) *If the feasible region for a linear programming problem is bounded, then both the maximum value and the minimum value of the objective function always exist.*

(B) *If the feasible region is unbounded and the coefficients of the objective function are positive, then the minimum value of the objective function exists, but the maximum value does not.*

(C) *If the feasible region is empty, then both the maximum value and the minimum value of the objective function do not exist.*

Geometric Method for Solving Linear Programming Problems.

Procedure (Geometric Method for Solving a Linear Programming Problem with Two Decision Variables).

(1) *Graph the feasible region. Then, if an optimal solution exists according to Theorem 2, find the coordinates of each corner point.*

(2) *Construct a* corner point table *listing the value of the objective function at each corner point.*

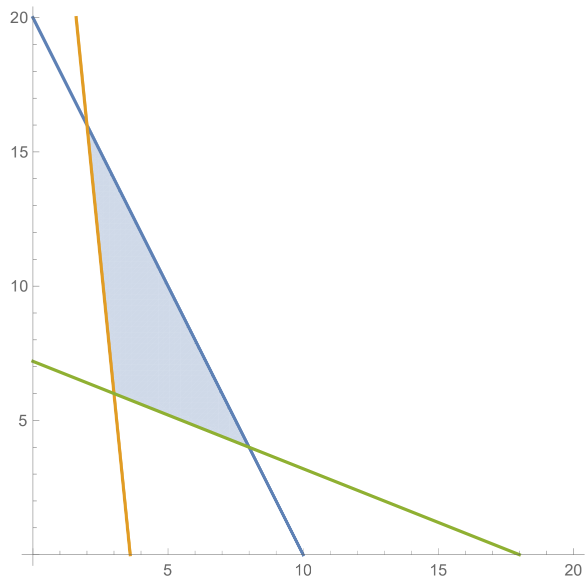
(3) *Determine the optimal solution(s) from the table in Step 2 (smallest=minimum, largest=maximum).*

(4) *For an applied problem, interpret the optimal solution(s) in terms of the original problem.*

Example 2. *Maximize and minimize $z = 3x + y$ subject to the inequalities*

$$\begin{aligned} 2x + y &\leq 20 \\ 10x + y &\geq 36 \\ 2x + 5y &\geq 36 \\ x, y &\geq 0 \end{aligned}$$

Solution. *We begin by graphing the feasible region*



Since this is a bounded region, by part (A) of Theorem 2, this problem has both a maximum and a minimum value. Now we find the corner points, of which there are three.

colors	*system*	*corner point*
blue and orange	$\begin{cases} 2x + y = 20 \\ 10x + y = 36 \end{cases}$	$(2, 16)$
blue and green	$\begin{cases} 2x + y = 20 \\ 2x + 5y = 36 \end{cases}$	$(8, 4)$
orange and green	$\begin{cases} 10x + y = 36 \\ 2x + 5y = 36 \end{cases}$	$(3, 6)$

Now, we check the value of z at these corner points:

corner point	$z = 3x + y$
$(2, 16)$	$z = 3(2) + (16) = 6 + 16 = 22$
$(8, 4)$	$z = 3(8) + (4) = 24 + 4 = 28$
$(3, 6)$	$z = 3(3) + (6) = 9 + 6 = 15$

Then, we see that the minimum value is 15 and occurs at $(3, 6)$ and the maximum value is 28 and occurs at $(8, 4)$.

Example 3. *Maximize and minimize* $z = 2x + 3y$ *subject to*

$$\begin{aligned} 2x + y &\geq 10 \\ x + 2y &\geq 8 \\ x, y &\geq 0 \end{aligned}$$

Solution. *Minimum of* $z = 14$ *at* $(4, 2)$. *No maximum.*

Example 4. *Maximize and minimize* $P = 30x + 10y$ *subject to*

$$\begin{aligned} 2x + 2y &\geq 4 \\ 6x + 4y &\leq 36 \\ 2x + y &\leq 10 \\ x, y &\geq 0 \end{aligned}$$

Solution. *Minimum of* $P = 20$ *at* $(0, 2)$. *Maximum of* $P = 150$ *at* $(5, 0)$.

Example 5. *Maximize and minimize* $P = 3x + 5y$ *subject to*

$$\begin{aligned} x + 2y &\leq 6 \\ x + y &\leq 4 \\ 2x + 3y &\geq 12 \\ x, y &\geq 0 \end{aligned}$$

Solution. *No optimal solutions.*